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When Is the Wave Function Single-Valued?

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It is shown that single-valuedness of the wave function can be lost because of an external field approximation. The Aharonov–Bohm effect is studied in detail as an example of the problem. Specifically, it is shown that the solenoid (represented as a rotating, charged cylinder) has a wave function that undergoes a phase shift equal in magnitude, but with opposite sign, to the phase shift suffered by the electron's wave function when the electron passes the solenoid.

1. INTRODUCTION

Single-valuedness of the wave function is one of the basic tenets of quantum theory. Yang (1983) has written, "We emphasize that to challenge the single valuedness of the wave function is to challenge the very foundation of quantum mechanics itself." The present authors are in complete agreement with this statement. However, the statement that the wave function must always be single-valued must be clearly understood. Rules that are used without understanding are generally misused. The goal of this paper is to shed some light on single-valuedness, and, above all, to point out one misuse of the single-valuedness rule.

In discussing single-valuedness, it is essential to distinguish between free particles and bound states. In bound states, the superposition principle by itself guarantees single-valuedness. How it does this has been illustrated clearly by Al-Jaber and Henneberger (1992). Again, we stress that in bound states, single-valuedness is a consequence of the superposition principle, not a separate fundamental requirement. Unfortunately, there have been several papers written using bound states to demonstrate the need for single-

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valuedness. We shall leave this subject at this point, since all workers agree that bound states must be single-valued.

The question of single-valuedness in the case of free particles now arises. Let us examine the case in which interacting particles are infinitely far apart at positive infinite and negative infinite times. Let us assume that a system consists of two interacting particles which do not interact with anything in the rest of the universe. Let the solution of the appropriate wave equation be given by $\Psi(\mathbf{r}_1, \mathbf{r}_2, t)$. Then, all workers, including the present authors, agree that a 360° rotation of the entire system about any axis must leave $\Psi(\mathbf{r}_1, \mathbf{r}_2, t)$ invariant. (We, of course, omit spinors from the discussion.) A 360° rotation of the entire universe is undetectable in any reasonable theory. The difficulty here is that the single-valuedness property of the wave function can be forfeited in making approximations. Quantum theory requires that the exact wave function depend upon the coordinates of both particles. However, in practice, when one of the particles is extremely massive, or perhaps not a particle at all, but a macroscopic object, it is common practice to assume that the small particle can have no influence on the large one. The larger (macroscopic) particle is then replaced by an external field. It is this external field approximation that causes the problem. The most notorious example of this is provided by the Aharonov–Bohm (AB) effect. Much has been written about this effect since the original paper of Aharonov and Bohm (1959). Nevertheless, misconceptions regarding this effect are commonplace even today.

In the following sections, it will be shown that in the AB effect the solenoid (which will be represented by a charged, rotating cylinder) also undergoes a phase shift. Moreover, this phase shift does not go to zero as the moment of inertia of the cylinder goes to infinity. The phase shift of the cylinder is of no consequence if one wishes information only about the cylinder. However, this phase shift is of vital importance in the discussion of the single-valuedness of $\Psi(\mathbf{r}, \theta, t)$, where θ represents the angle that locates some fiducial mark on the cylinder. In the usual external field approximation, it is assumed that the passage of the electron can have no influence on the rotating cylinder. One thus writes

$$\Psi(\mathbf{r}, \theta, t) = \Theta(\theta, t)\psi(\mathbf{r}, t) \tag{1}$$

and assumes that $\Theta(\theta, t)$ can be ignored and that $\psi(\mathbf{r}, t)$ then is the "wave function" for the electron in the external field. One then applies the single-valuedness rule to $\psi(\mathbf{r}, t)$. In the remainder of this work, it will be shown that equation (1) is incorrect, since Θ must depend upon $\mathbf{r}$ as well as θ and t. It will also be shown that $\psi(\mathbf{r}, t)$ cannot, in general, be single-valued.

2. CLASSICAL EQUATION FOR AN ELECTRON IN AN EXTERNAL FIELD

From a classical point of view, the external field approximation is a good one. It may therefore be helpful to the reader to review the work of Zhu and Henneberger (1990) at this point.

The interaction energy of a time-independent external current with a passing electron is given by

$$\Delta E = \frac{1}{4\pi} \int \mathbf{B}_{\text{el}}(\mathbf{r}' - \mathbf{r}) \cdot \mathbf{B}_{\text{ext}}(\mathbf{r}')\, d^3\mathbf{r}' \tag{2}$$

with

$$\mathbf{B}_{el} = \frac{1}{c}\, \mathbf{v} \times \mathbf{E}(\mathbf{r}' - \mathbf{r}) \tag{3}$$

where $\mathbf{v}$ is the velocity of an electron at the point $\mathbf{r}$. The reader will note that the electrostatic energy

$$\Delta E_{\text{el}} = \frac{1}{4\pi} \int \mathbf{E}_{\text{el}}(\mathbf{r}' - \mathbf{r}) \cdot \mathbf{E}_{\text{ext}}(\mathbf{r}')\, d^3\mathbf{r}' \tag{4}$$

vanishes if the point $\mathbf{r}$ is a point at which the electrostatic potential vanishes. This is the case in the example to be considered. We consider the quantity

$$\begin{aligned}
&\frac{1}{4\pi} \nabla \int \mathbf{B}_{\text{el}}(\mathbf{r}' - \mathbf{r}) \cdot \mathbf{B}_{\text{ext}}(\mathbf{r}')\, d^3\mathbf{r}' \\
&= \frac{1}{4\pi c} \nabla \int \mathbf{v} \times \mathbf{E}(\mathbf{r}' - \mathbf{r}) \cdot \mathbf{B}_{\text{ext}}(\mathbf{r}')\, d^3\mathbf{r}' \\
&= \frac{1}{4\pi c} \nabla \int \mathbf{v} \cdot \mathbf{E}(\mathbf{r}' - \mathbf{r}) \times \mathbf{B}_{\text{ext}}(\mathbf{r}')\, d^3\mathbf{r}' \\
&= \nabla(\mathbf{v} \cdot \mathbf{P}_{\text{field}}) \\
&= (\mathbf{v} \cdot \nabla)\mathbf{P}_{\text{field}} + \mathbf{v} \times (\nabla \times \mathbf{P}_{\text{field}}) \\
&= \frac{d\mathbf{P}_{\text{field}}}{dt} + \mathbf{v} \times (\nabla \times \mathbf{P}_{\text{field}})
\end{aligned} \tag{5}$$

Equation (5) makes use of the expression for the electromagnetic momentum, which depends upon the electric field of the electron and the magnetic field of the solenoid. The second term on the right-hand side of equation (5) is just $(e/c)\mathbf{v}\times\mathbf{B}_{\text{ext}}$. The proof is as follows:

$$\begin{aligned}\frac{1}{4\pi c}\nabla\times\int \mathbf{E}(\mathbf{r}'-\mathbf{r})\times\mathbf{B}_{\text{ext}}(\mathbf{r}')\,d^3\mathbf{r}'\\ =\frac{1}{4\pi c}\bigg\{-\mathbf{B}_{\text{ext}}(\mathbf{r}')[\nabla\cdot\mathbf{E}(\mathbf{r}'-\mathbf{r})]\,d^3\mathbf{r}'\\ +\int[\mathbf{B}_{\text{ext}}(\mathbf{r}')\cdot\nabla]\mathbf{E}(\mathbf{r}'-\mathbf{r})\,d^3\mathbf{r}'\bigg\}\end{aligned}\tag{6}$$

The relation $\nabla\cdot\mathbf{E}(\mathbf{r}'-\mathbf{r})=-\nabla'\cdot\mathbf{E}(\mathbf{r}'-\mathbf{r})$ yields

$$\begin{aligned}\nabla\times\mathbf{P}_{\text{field}}=\frac{1}{4\pi c}\bigg\{\int\mathbf{B}(\mathbf{r}')\cdot 4\pi e\delta(\mathbf{r}'-\mathbf{r})\,d^3\mathbf{r}'\\ +\int[\mathbf{B}(\mathbf{r}')\cdot\nabla]\mathbf{E}(\mathbf{r}'-\mathbf{r})\,d^3\mathbf{r}'\bigg\}\end{aligned}\tag{7}$$

The first term of equation (7) is $(e/c)\mathbf{B}(\mathbf{r})$. The second term can be shown to vanish. Let $\boldsymbol{\varepsilon}$ be an arbitrary constant vector. Then the relation

$$\begin{aligned}\boldsymbol{\varepsilon}\cdot\int[\mathbf{B}(\mathbf{r}')\cdot\nabla']\mathbf{E}(\mathbf{r}'-\mathbf{r})\,d^3\mathbf{r}'\\ =\int[\mathbf{B}(\mathbf{r}')\cdot\nabla'][\boldsymbol{\varepsilon}\cdot\mathbf{E}(\mathbf{r}'-\mathbf{r})]\,d^3\mathbf{r}'\\ =\int\nabla'\cdot[\boldsymbol{\varepsilon}\cdot\mathbf{E}(\mathbf{r}'-\mathbf{r})]\mathbf{B}(\mathbf{r}')\,d^3\mathbf{r}'\\ -\int\boldsymbol{\varepsilon}\cdot\mathbf{E}(\mathbf{r}'-\mathbf{r})\,\nabla'\cdot\mathbf{B}(\mathbf{r}')\,d^3\mathbf{r}'\end{aligned}\tag{8}$$

holds. The first integral can be converted into a surface integral that vanishes as the surface goes to infinity. The second integral vanishes since $\nabla'\cdot\mathbf{B}(\mathbf{r}')=0$. Thus, we have the result

$$\frac{1}{4\pi}\nabla\int\mathbf{B}(\mathbf{r}'-\mathbf{r})\cdot\mathbf{B}_{\text{ext}}(\mathbf{r}')\,d^3\mathbf{r}'=\frac{d\mathbf{P}_{\text{field}}}{dt}+\frac{e}{c}\mathbf{v}\times\mathbf{B}_{\text{ext}}\tag{9}$$

The reader will observe that equation (9) assumes that the external field $\mathbf{B}_{\text{ext}}$ is absolutely constant in time and not influenced by the moving electron. This assumption is the one made in almost all treatments of the AB effect. This condition can be fulfilled easily to arbitrary accuracy by ensuring that the solenoid has a sufficiently large (mechanical or electromagnetic) moment of inertia.

Equations (6)–(9) yield the result

$$\nabla\times\mathbf{P}_{\text{field}}=(e/c)(\nabla\times\mathbf{A})\tag{10}$$

This indicates that, in some gauge,

$$\mathbf{P}_{\text{field}}(\mathbf{r}) = (e/c)\mathbf{A}(\mathbf{r}) \tag{11}$$

It should not be surprising to learn that the gauge in question is the Coulomb gauge, i.e., the gauge satisfying $\nabla \cdot \mathbf{A} = 0$. This result was already obtained by Boyer (1973). The significance of Boyer's result seems never to have been fully appreciated. The physical (observable) quantity responsible for the AB effect is the electromagnetic momentum. Gauge invariance of the theory causes any effect of the electromagnetic momentum to appear to be an effect of the vector potential. Modulo a gauge transformation, one might consider $(e/c)\mathbf{A}(\mathbf{r})$ to represent the electromagnetic momentum in an arbitrary gauge.

The following is a proof of equation (11) due to Zhu and Henneberger (1990):

$$\begin{aligned}
\mathbf{P}_{\text{field}}(\mathbf{r}) &= \frac{1}{4\pi c}\int \mathbf{E}(\mathbf{r}'-\mathbf{r}) \times [\nabla' \times \mathbf{A}(\mathbf{r}')\, d^3\mathbf{r}' \\
&= \frac{1}{4\pi c}\int \nabla'[\mathbf{E}(\mathbf{r}'-\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}')]\, d^3\mathbf{r}' \\
&\quad - \frac{1}{4\pi c}\int [\mathbf{A}(\mathbf{r}') \cdot \nabla']\mathbf{E}(\mathbf{r}'-\mathbf{r})\, d^3\mathbf{r}' \\
&\quad - \frac{1}{4\pi c}\int [\mathbf{E}(\mathbf{r}'-\mathbf{r}) \cdot \nabla']\mathbf{A}(\mathbf{r}')\, d^3\mathbf{r}' \\
&\quad - \frac{1}{4\pi c}\int \mathbf{A}(\mathbf{r}') \times [\nabla' \times \mathbf{E}(\mathbf{r}'-\mathbf{r})]\, d^3\mathbf{r}'
\end{aligned} \tag{12}$$

The fourth term of equation (12) vanishes, since $\nabla' \times \mathbf{E}(\mathbf{r}'-\mathbf{r}) = 0$ for the Coulomb field of the electron. The first integral vanishes by a corollary of Gauss' theorem. The second integral vanishes in Coulomb gauge: Let $\boldsymbol{\varepsilon}$ be an arbitrary constant vector. We then have the equality

$$\begin{aligned}
\boldsymbol{\varepsilon} \cdot \int [A(\mathbf{r}') \cdot \nabla']\mathbf{E}(\mathbf{r}'-\mathbf{r})\, d^3\mathbf{r}' &= \int \nabla' \cdot [\mathbf{A}(\mathbf{r}')\boldsymbol{\varepsilon} \cdot \mathbf{E}(\mathbf{r}'-\mathbf{r})]\, d^3\mathbf{r}' \\
&\quad - \int [\boldsymbol{\varepsilon} \cdot \mathbf{E}(\mathbf{r}'-\mathbf{r})]\, \nabla' \cdot \mathbf{A}(\mathbf{r}')\, d^3\mathbf{r}'
\end{aligned} \tag{13}$$

The first integral can be converted into a vanishing surface integral. The second term vanishes because of $\nabla' \cdot \mathbf{A}(\mathbf{r}') = 0$.

The third integral of equation (12) can be written

$$(1/4\pi c)\int \mathbf{A}(\mathbf{r}')\,\nabla'\cdot\mathbf{E}(\mathbf{r}'-\mathbf{r})\,d^3\mathbf{r}'$$

$$=(1/c)\int \mathbf{A}(\mathbf{r}')e\,\delta(\mathbf{r}'-\mathbf{r})\,d^3\mathbf{r}'=(e/c)\mathbf{A}(\mathbf{r}) \tag{14}$$

Therefore, in a realistic problem involving sources of finite extent in space, the electromagnetic momentum is given by $(e/c)\mathbf{A}(\mathbf{r})$, where $\mathbf{A}(\mathbf{r})$ is the (unique) vector potential in Coulomb gauge.

3. THE FREE-PARTICLE AHARONOV–BOHM EFFECT

The essential feature of the Aharonov–Bohm effect is that the wave function of an electron confined to a region external to a whisker of flux undergoes a phase shift even though no force acts upon it. The sign of the phase shift depends upon the sign of the kinetic angular momentum. This path-dependent phase shift results in a shift in electron diffraction patterns when a flux whisker is introduced into the diffracting system. Definitive experimental work has been done by Chambers (1960), Boersch *et al.* (1962), Möllenstedt and Bayh (1962), and more recently by Tonomura and co-workers (1986). This shift in a diffraction pattern is often called "Aharonov–Bohm scattering." Equation (9) indicates that the electromagnetic momentum is the quantity being scattered.

Aharonov and Bohm (1959) computed a scattering cross section for electrons on a vanishingly small-diameter flux whisker. The "scattering" has the distressing property that a vanishing force (i.e., a force having zero range) gives rise to an infinite total scattering cross section.

The usual treatment of the problem consists in going immediately to the external field approximation. This is usually done under the assumption that the result obtained is still exact, at least for all practical purposes. There exist two sets of angular momentum eigenfunctions that may be considered serious candidates for the designation "stationary states of the system." For motion confined to the x, y plane, these are:

1. $J_m(kr)\,e^{im\phi}\,e^{-i\alpha\phi}$
2. $J_{|m+\alpha|}(kr)\,e^{im\phi}$

In the above, m is an integer and $\alpha = -(e\Phi/ch)$, where Φ is the flux in a whisker along the z axis.

It was shown by Dirac (1931) that in a region free of magnetic fields the solution of the Schrödinger equation may be written

$$\psi(\mathbf{r},t)=\psi^0(\mathbf{r},t)\exp\left[i(e/\hbar c)\int^{\mathbf{r}}\mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}'\right] \tag{15}$$

where $\psi^0(\mathbf{r}, t)$ satisfies the Schrödinger equation with the same scalar potential, but in which the vector potential has been set equal to zero. The vector potential for a flux whisker is

$$\mathbf{A}(\mathbf{r}) = \hat{\phi}\Phi/(2\pi\rho) \tag{16}$$

Thus,

$$\begin{aligned}\int^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}' &= \int^{\phi} \frac{\Phi\rho'}{2\pi\rho'}\, d\phi' \\ &= \frac{\Phi\phi}{2\pi} + \text{const}\end{aligned} \tag{17}$$

and the phase factor of equation (15), which is called the Dirac phase factor, is (apart from a constant phase) just the factor $e^{-i\alpha\phi}$ of the solutions of type 1. Hereafter, we refer to any single-valued function multiplied by the Dirac phase factor as a Dirac wave function.

The functions of type 2 were first obtained by Aharonov and Bohm (1959). These functions follow as a consequence of a routine application of the single-valuedness rule. These functions, as well as their superpositions, will be called AB wave functions.

4. THE COMPLETE AB PROBLEM

A discussion of the AB effect when the flux whisker is an actual solenoid is extremely complicated. The reader is referred to Sections 5 and 6 of the work of Zhu and Henneberger (1990). In order to avoid a discussion of the quantum theory of the power source that supplies the current to the solenoid, we adopt a model similar to the one discussed in Section 5 of Peshkin *et al.* (1961).

We consider a cylinder of radius a and length l carrying a surface charge density σ. The axis of the cylinder carries a line charge $\lambda = -2\pi a\sigma$, so that the electric field of the cylinder is confined to its interior. Thus, the electrostatic potential vanishes in the region exterior to the cylinder. The cylinder is free to rotate about its axis (the z axis).

An exact treatment of the AB problem in which an electron passes the cylinder with fixed angular momentum consists of an infinite series of rapidly converging corrections to the unperturbed motion. The passing electron affects the motion of the cylinder, the change in the cylinder's motion gives a further correction to the usual AB effect, etc.

The method that we use here is an approximation, albeit an extremely good one. We approximate the state function for the electron–cylinder system by a product of the form

$$\Psi(\mathbf{r}, \theta, t) = \Theta(\theta, \mathbf{r}, t)\psi(\mathbf{r}, t) \tag{18}$$

The reader will note that this differs from equation (1) only in the fact that $\Theta(\theta, \mathbf{r}, t)$ now depends on $\mathbf{r}$, in order to allow for the influence of the electron on the cylinder. We reiterate: It is $\Psi(\mathbf{r}, \theta, t)$ which describes an isolated system. It is *this* function which must be single-valued. The product ansatz of equation (18) is based on the fact that the influence of the passing electron on the cylinder (and vice versa!) is very slight.

We obtain an approximate solution as follows:

1. We assume that $\psi(\mathbf{r}, t)$ is a solution of the usual AB problem using a *constant* flux.

2. We assume that $\Theta(\theta, \mathbf{r}, t)$ is the solution of the quantum problem for the cylinder interacting with a passing classical electron. This may appear strange, but in light of the fact that the electron experiences no force, the approximation should be excellent.

We consider that a classical electron passes the cylinder with constant speed along a line parallel to the x axis with impact parameter b, as shown in Fig. 1.

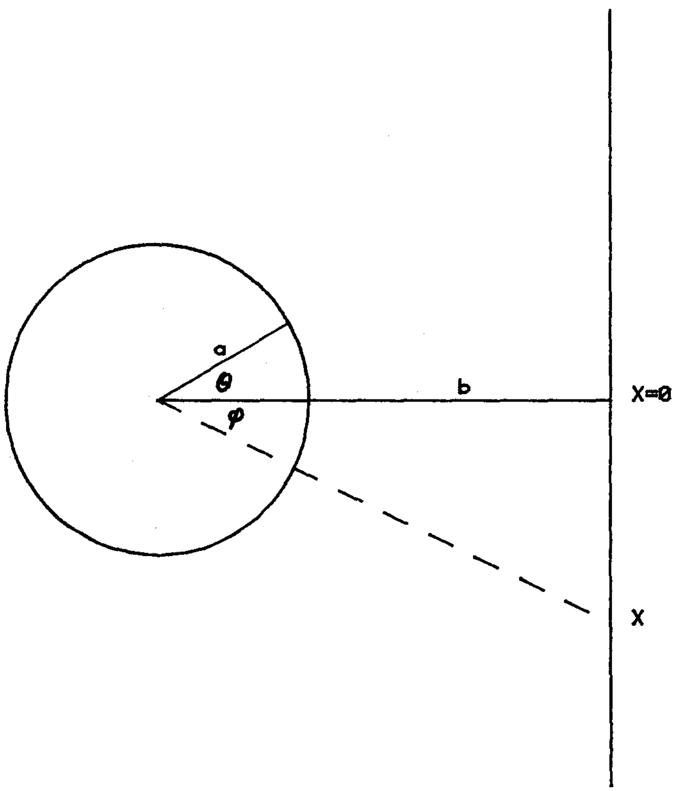

Fig. 1.

At this point, we again remind the reader that the form of $\psi(\mathbf{r}, t)$, i.e., whether $\psi(\mathbf{r}, t)$ is a Dirac wave function or a single-valued wave function, depends upon our result for $\Theta(\theta, \mathbf{r}, t)$.

Equations (9) and (11) together yield

$$\frac{1}{4\pi} \nabla \int \mathbf{B}_{\text{el}}(\mathbf{r}' - \mathbf{r}) \cdot \mathbf{B}_{\text{cyl}}(\mathbf{r}')\, d^3\mathbf{r}' = \frac{e}{c} \frac{d\mathbf{A}(\mathbf{r})}{dt} \tag{19}$$

Equation (19) holds only in Coulomb gauge; the Lorentz force term, which vanishes in the AB problem, has been dropped. $\mathbf{B}_{\text{cyl}}(\mathbf{r}')$ is in the z direction. The magnitude of the magnetic field is related to the current/length K by the relation $B_{\text{cyl}} = 4\pi K/c$, with $K = \dot{\theta} a\sigma$. Here, $\dot{\theta}$ is the angular velocity of the cylinder, so that

$$B_{\text{cyl}} = (4\pi/c)\dot{\theta} a\sigma \tag{20}$$

Equation (19), together with the fact that the electron has only one degree of freedom, yields

$$\begin{aligned} &\frac{\partial}{\partial \mathbf{x}} \int_{\text{Vol}} \frac{\dot{\theta} a\sigma}{c} B_z(\mathbf{r}' - \mathbf{r})\, d^3\mathbf{r}' \\ &\quad = \frac{\partial}{\partial \mathbf{x}} \int_{\text{Vol}} \frac{\dot{\theta} a\sigma}{c} \Phi_{\text{el}}(z', x)\, dz' \\ &\quad = \frac{e}{c} \frac{dA_x(\mathbf{r})}{dt} \\ &\quad = \frac{e}{c} v \frac{\partial A_x}{\partial x} \end{aligned} \tag{21}$$

In equation (21), Vol denotes the volume of the cylinder, and v is the velocity of the electron. x and t are related by $x = vt$. The constancy of v assumes the neglect of electric fields caused by the angular acceleration of the cylinder. In this approximation (which is excellent to a massive cylinder with a large angular momentum), one may integrate equation (21) directly, obtaining

$$(e/c)vA_x = (\dot{\theta} a\sigma/c) \int_{\text{Vol}} \Phi_{\text{el}}(z', x)\, dz' \tag{22}$$

The cylinder has length l and (mechanical) moment of inertia I_0. The electromagnetic angular momentum of the cylinder is

$$\mathbf{L}_{\text{em}} = \frac{1}{4\pi c} \int_{\text{Vol}} \mathbf{r} \times (\mathbf{E} \times \mathbf{B})\, d^3\mathbf{r} \tag{23}$$

with $\mathbf{r} = \mathbf{i}x + \mathbf{j}y$. The $\mathbf{L}_{\text{em}}$ is parallel to $\mathbf{L}_{\text{mech}}$. We have

$$\mathbf{L}_{\text{em}} = (l/4\pi c)\int_0^a r(4\pi a\sigma/r)(4\pi/c)\dot{\theta}a\sigma \cdot 2\pi r\,dr = 4\pi^2 a^4\sigma^2 l\dot{\theta}/c^2 \tag{24}$$

so that $I = I_0 + 4\pi^2 a^4\sigma^2 l/c^2$ is the effective moment of inertia of a charged cylinder of length l.

The interaction Lagrangian is given by $(e/c)\mathbf{v}\cdot\mathbf{A}$, so that the Lagrangian for the cylinder is

$$\mathscr{L} = \frac{1}{2}I\dot{\theta}^2 + \frac{e}{c}vA_x \tag{25}$$

Here, θ is the angle subtended by some fiducial mark on the cylinder with the perpendicular to the trajectory indicated by the vector $\mathbf{b}$ in Fig. 1. The vector potential is $\mathbf{A} = \hat{\phi}\Phi/(2\pi r)$, where Φ is the flux due to the rotating cylinder [not to be confused with Φ_{el} of equation (22)]. The unit vector $\hat{\phi}$ is in the ϕ direction in Fig. 1.

The flux is given by

$$\Phi = \pi a^2 B = (4\pi^2 a^3\sigma/c)\dot{\theta} \tag{26}$$

and

$$A_\phi = \frac{2\pi a^3\sigma}{c(b^2+x^2)^{1/2}}, \qquad \cos\phi = \frac{b}{(b^2+x^2)^{1/2}} \tag{27a}$$

$$A_x = A_\phi \cos\phi = \frac{2\pi a^3\sigma b\dot{\theta}}{c(b^2+x^2)} \tag{27b}$$

yield

$$\mathscr{L} = \frac{1}{2}I\dot{\theta}^2 + \frac{2\pi e a^3\sigma b v\dot{\theta}}{c^2(b^2+x^2)} \tag{28}$$

The canonical angular momentum is

$$P_\theta = \partial\mathscr{L}/\partial\dot{\theta} = I\dot{\theta} + R(x), \qquad R(x) = \frac{2\pi e a^3\sigma b v}{c^2(b^2+x^2)} \tag{29}$$

with $x = vt$. The relation $\partial\mathscr{L}/\partial\theta = 0$ indicates that P_θ is conserved. The wave function $\Theta(\theta, \mathbf{r}, t)$ of equation (18) is of the form

$$\Theta(\theta, \mathbf{r}, t) = \exp\left\{\frac{i}{\hbar}\left[\int_{-\infty}^{\theta} P_\theta\,d\theta' - \int_{-\infty}^{t} E(t')\,dt'\right]\right\} \tag{30}$$

The interaction between the electron and the cylinder is very weak. We may therefore consider that the total energy of the system comprises that of the electron plus that of the cylinder. The energy of the electron (kinetic energy $= \frac{1}{2}mv^2$) is conserved. Energy conservation therefore dictates that the energy of the cylinder (mechanical plus magnetic) is conserved. This is easily checked. We define the quantity ω:

$$P_\theta \equiv I\omega = I\dot{\theta}\big|_{t=-\infty} \tag{31}$$

Then equation (29) yields $\dot{\theta} = \omega - R(x)/I$, so that

$$\Delta\dot{\theta} = -R(x)/I \tag{32}$$

represents the change in $\dot{\theta}$ due to the passage of the electron. The change in kinetic energy is then

$$\Delta \mathrm{KE} = I\dot{\theta}\,\Delta\dot{\theta} \approx I\omega\,\Delta\dot{\theta} = -\frac{2\pi e a^3 \sigma b v \omega}{c^2(b^2+x^2)} \tag{33}$$

The change in magnetic energy is

$$\begin{aligned}
&\frac{1}{4\pi}\int \mathbf{B}_{\mathrm{el}}(\mathbf{r}'-\mathbf{r})\cdot\mathbf{B}_{\mathrm{cyl}}(\mathbf{r}')\,d^3\mathbf{r}' \\
&= \frac{\omega a \sigma}{c}\int \Phi_{\mathrm{el}}(z',x)\,dz' \\
&= \frac{e}{c}\,vA_x \\
&= \frac{e}{c^2}\frac{2\pi a^3 \sigma b v}{b^2+x^2}\,\omega \\
&= -\Delta \mathrm{KE}
\end{aligned} \tag{34}$$

It is now clear that any phase shift of the cylinder must be due to a lag in the angle $\theta(t)$. Indeed, the phase shift in equation (30) is given by

$$\begin{aligned}
\frac{1}{\hbar}P_\theta\,\Delta\theta &= \frac{I\omega}{\hbar}\int_{-\infty}^{t}\Delta\dot{\theta}\,dt' \\
&= -\frac{\omega}{\hbar}\int_{-\infty}^{x}\frac{2\pi e a^3\sigma b}{c^2(b^2+x'^2)}\,dx' \\
&= -\frac{e}{\hbar c}\int_{-\infty}^{x} A_x\,dx'
\end{aligned} \tag{35}$$

Thus, the phase shift of $\Theta(\theta, \mathbf{r}, t)$ is just the negative of the Dirac phase shift that a quantum electron experiences in the vector potential of a classical flux.

We now return to the requirement that the total wave function $\Psi(\mathbf{r}, \theta, t) = \Theta(\theta, \mathbf{r}, t)\psi(\mathbf{r}, t)$ must be single-valued. The phase factor occurring in $\Theta(\theta, \mathbf{r}, t)$ requires that $\psi(\mathbf{r}, t)$ contain the Dirac phase factor discussed earlier. The use of single-valued functions for $\psi(\mathbf{r}, t)$ violates the single-valuedness condition for the overall wave function $\Psi(\mathbf{r}, \theta, t)$.

5. DISCUSSION

We have seen how the external field approximation can destroy the single-valuedness requirement of wave functions of particles interacting with such fields. An object may have macroscopic dimensions and still carry a phase factor in its wave function that profoundly affects the boundary conditions for a quantum particle with which it interacts. One may not, in general, blindly apply a single-valuedness condition to external field problems unless one is dealing with bound states. Workers dealing with external field problems who wish to avoid the quantum theory of the external source may find the use of boundary conditions involving the probability current density, as discussed elsewhere (Henneberger, 1984), helpful.

Finally, we note that the appropriate stationary states in the AB "scattering" problem are the Dirac states, not the AB states. This is a point that one of the authors has stressed for over a decade (Henneberger, 1981, 1984; Shapiro and Henneberger, 1989).

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